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A Microscopic Model of Interface Related to the Burgers Equation

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A microscopic model for a solid-on-solid type of interface under the influence of an external field is introduced. It is proven that in equilibrium the macroscopic profile satisfies a partial differential equation which is (up to a transformation) the stationary Burgers equation. The study is based on the structure of the invariant measures for a related asymmetric simple exclusion process.

KEY WORDS: Interface models; solid-on-solid interface; Burgers equation; asymmetric simple exclusion process.

1. INTRODUCTION

In recent years there has been considerable interest in surfaces growing through deposition and in growth patterns in cluster solidification fronts. In this paper we present a very simple stochastic model which describes the time evolution of an interface, of type "solid on solid," under the influence of an external field in a strip of width L. In a properly chosen continuum limit (lattice spacing $1/L$, $L \to \infty$), and in equilibrium, we obtain that the macroscopic height $h(\cdot)$ of the interface satisfies the equation

$$\frac{\partial^2 h}{\partial r^2} = \frac{1}{2}\left[\left(\frac{\partial h}{\partial r}\right)^2 - 1\right] \tag{1.1a}$$

$$h(0) = h(1) = 0 \tag{1.1b}$$

Equation (1.1) is related to the stationary Burgers equation by the transformation

$$h' = \rho \tag{1.2}$$

We dedicate this paper to our friend Paola Calderoni.

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The Burgers equation describes also the dynamics of one-dimensional, weakly asymmetric lattice gases on a macroscopic scale. We refer to ref. 1 for a detailed analysis. Our study is based on a simple relationship at the microscopic level between the interface model and a lattice gas (simple exclusion process). For this model we prove, in a suitable scaling, that the stationary measure has on the global scale the density ρ of particles and is on a local scale homogeneous and independent.

The relationship between lattice gases and interfaces has also been studied by Gallavotti[6] for the Ising model in a box. Rost[10] related the totally asymmetric simple exclusion process with the boundary of an Eden model[2] for a time-dependent situation. For a review of the symmetric solid-on-solid model see ref. 5. Kardar *et al.*[7] have studied ballistic deposition on surfaces. They noted and used the relationship (1.2) to study the Burgers equation with external random force.

In Section 2 we introduce the interface model, the asymmetric simple exclusion, and the isomorphism relating them. Then we justify the rescaling of the drift as a function of L (weak asymmetry), and state our results: Theorem 1 contains the statements of local homogeneity and independence for the stationary measure of the simple exclusion model. Theorem 2 states the convergence to the macroscopic profile of the interface model in equilibrium. The proofs are given in Section 3. Finally, in Section 4 we make a few comments on the fluctuations around the deterministic limit of Theorem 1.

2. DEFINITION OF THE MODEL. STATEMENT OF THE RESULTS

Definition 2.1. *The interface model.* Let $L \geqslant 1$ be an integer and $\Lambda(L) \stackrel{\text{def}}{=} \{x \in \mathbb{Z}\colon 0 \leqslant x \leqslant L\}$. The interface model we study is the Markov process $(\xi_t^L)_{t \geqslant 0}$ taking values on

$$\mathscr{X}_L \stackrel{\text{def}}{=} \{\xi \in \mathbb{Z}^{\Lambda(2L)}\colon \xi(0) = \xi(2L) = 0,\ |\xi(x) - \xi(x+1)| = 1, \text{for } 0 \leqslant x \leqslant 2L-1\}$$

and whose generator $\mathscr{L}_t^L$ is given by

$$\begin{aligned}\mathscr{L}_t^L f(\xi) = \sum_{1 \leqslant x \leqslant 2L-1} &\{p1\{\xi(x-1) = \xi(x+1) < \xi(x)\}[f(\xi^{x,-}) - f(\xi)] \\ &+ q1\{\xi(x-1) = \xi(x+1) > \xi(x)\}[f(\xi^{x,+}) - f(\xi)]\}\end{aligned} \tag{2.1}$$

where $1\{\cdot\}$ is the characteristic function of the set $\{\cdot\}$, f is a real function defined on $\mathscr{X}_L$, $0 < p < 1$, $q = 1 - p$, and $\xi^{x,-}$, $\xi^{x,+}$ are given by

$$\xi^{x,\pm}(y) = \begin{cases} \xi(y) & \text{if } \ y \neq x \\ \xi(y) \pm 2 & \text{if } \ y = x \end{cases} \tag{2.2}$$

The generator $\mathscr{L}_i^L$ governs an interface dynamics which can be informally described as follows: the height of the interface at a given site, say x, can change only if heights at its nearest neighbor sites are both equal. In this case, with rate q, it changes by two units up if $\xi(x) < \xi(x-1)$, and with rate p, it changes by two units down if $\xi(x) > \xi(x-1)$. By symmetry it suffices to consider the case $p \geqslant 1/2$.

Definition 2.2. *The asymmetric simple exclusion model.* Let $(\eta_t^L)_{t \geqslant 0}$ be the Markov process with state space $\mathscr{Y}_L = \{\eta \in \{0, 1\}^{A(2L+1)}: \eta(0) = \eta(2L+1) = 1\}$ and generator $\mathscr{L}_e^L$ given by $(1/2 \leqslant p < 1,\ q = 1 - p)$

$$\begin{aligned}\mathscr{L}_e^L f(\eta) = \sum_{1 \leqslant x \leqslant 2L} \{&p\eta(x)[1-\eta(x+1)][f(\eta_{x,x+1}) - f(\eta)] \\ &+ q\eta(x)[1-\eta(x-1)][f(\eta_{x,x-1}) - f(\eta)]\}\end{aligned} \tag{2.3}$$

for $f: \mathscr{Y}_L \to \mathbb{R}$ and where $\eta_{x,y}$ is the configuration given by

$$\eta_{x,y}(z) = \begin{cases} \eta(z) & \text{if} \quad z \neq x, y \\ \eta(x) & \text{if} \quad z = y \\ \eta(y) & \text{if} \quad z = x \end{cases} \tag{2.4}$$

for $1 \leqslant x,\ y \leqslant 2L$.

We say that site x is empty if $\eta(x) = 0$ and that there is one particle at x if $\eta(x) = 1$. Informally, the generator $\mathscr{L}_i^L$ describes the motion of particles with the following behavior. Each particle jumps with rate p (respectively q) to its nearest right (left) unoccupied neighbor (jumps to occupied sites are forbidden). The condition that sites 0 and $2L+1$ are always occupied with particles that do not move can be interpreted as reflecting boundary conditions. This condition implies that the total current is identically zero. The extremal invariant measures for η_t^L form a family μ_n^L, $0 \leqslant n \leqslant 2L$, indexed by the number of particles in $\{1,..., 2L\}$, which is a conserved quantity. Under μ_n^L the occupation numbers at distinct sites are correlated. Nevertheless, it is easy to check that the product measures ν_α^L, $0 < \alpha < +\infty$, with marginals given by

$$\nu_\alpha^L(\eta(x) = 1) = \frac{\alpha(p/q)^x}{1 + \alpha(p/q)^x}, \qquad 1 \leqslant x \leqslant 2L \tag{2.5}$$

are reversible for the generator $\mathscr{L}_e^L$ (see also ref. 8). These are not extremal, but defining $A_n = \{\eta: \sum_{x=1}^{2L} \eta(x) = n\}$, the following is true:

$$\begin{aligned}&\mu_n^L = \nu_\alpha^L(\cdot \mid A_n) \qquad \text{for any} \quad \alpha > 0 \\ &\nu_\alpha^L = \sum_{n=0}^{2L} \nu_\alpha^L(A_n)\mu_\alpha^L\end{aligned} \tag{2.6}$$

To avoid heavy notation, we omit the superscript L on (η_t^L), μ_n^L, etc., when no confusion arises. In the particular case $n = L$ we write $\bar{\mu}_L$ $(= \mu_L^L)$.

Correspondence between the interface model and the asymmetric exclusion process. Let $\bar{\mathscr{Y}}_L = \{\eta \in \mathscr{Y}_L: \sum_{x=0}^{2L} \eta(x) = L\}$ and let H: $\xi \mapsto \eta$ be the bijection between $\mathscr{X}_L$ and $\bar{\mathscr{Y}}_L$ defined by

$$\begin{aligned} &\eta(x) = [1 + \xi(x) - \xi(x-1)]/2 \qquad \text{for} \quad 1 \leqslant x \leqslant 2L \\ &\eta(0) = \eta(2L+1) = 1 \end{aligned} \tag{2.7a}$$

Thus, $H^{-1}(\eta) = \xi$ is defined by

$$\begin{aligned} &\xi(x) = \sum_{y=1}^{x} [2\eta(y) - 1], \qquad \text{for} \quad 1 \leqslant x \leqslant 2L \\ &\xi(0) = 0 \qquad \text{and} \qquad \xi(2L) = 0 \end{aligned} \tag{2.7b}$$

From the previous definitions we conclude the following result.

Remark. For any $\eta \in \bar{\mathscr{Y}}_L$, if we let $(\eta_t)_{t \geqslant 0}$ be any regular version of the exclusion process (according to Definition 2.3), with $\eta_0 = \eta$, then $\xi_t = H^{-1}(\eta_t)$, $t \geqslant 0$, describes a regular version of the interface model (same p) defined by Definition 2.1, with $\xi_0 = H^{-1}(\eta)$, and conversely.

From the above remark it follows that the unique invariant probability for (ξ_t^L) is $\tilde{\mu}_L$ given by

$$\tilde{\mu}_L(A) = \bar{\mu}_L\{\eta \in \bar{\mathscr{Y}}_L: \ H^{-1}\eta \in A\}$$

for $A \subseteq \mathscr{X}_L$.

Up to now we have taken L and p fixed. Our interest is to study the behavior of $\tilde{\mu}_L$ (or $\bar{\mu}_L$) as $L \to \infty$. We take $r = x/(2L)$, $0 \leqslant r \leqslant 1$, and $p = p(L) = 1/2 + \theta/4L$ for some $\theta \neq 0$ in order to obtain a nontrivial limit, as will be explained later. We prove the following result for the asymmetric simple exclusion process.

Theorem 1. For $L \geqslant 2$, integer, let (η_t) be the exclusion process on $\bar{\mathscr{Y}}_L$ defined according to Definition 2.3 with $p = p(L) = 1/2 + \theta/4L$ for some $\theta \neq 0$ fixed. Let $\bar{\mu}_L$ $(= \mu_L^L)$ be its unique invariant probability. For any $r \in (0, 1)$ and any f cylinder function on $\{0, 1\}^Z$,

$$\lim_{L \to \infty} \bar{\mu}_L(f \cdot \tau_{[2rL]}) = \beta_{\rho(r)}(f) \tag{2.8}$$

Here β_ρ is the Bernoulli measure on $\{0, 1\}^Z$ with parameter ρ, i.e., β_ρ is the product probability on $\{0, 1\}^Z$, so that $\beta_\rho(\eta(x)=1)=\rho$ for all integer x; the parameter $\rho(r)$ is given by

$$\rho(r)=\frac{\exp\theta(2r-1)}{1+\exp\theta(2r-1)}$$

and satisfies

$$\frac{d\rho}{dr}=2\theta\rho(1-\rho), \qquad 0<r<1 \tag{2.9a}$$

$$\rho(0)=1-\rho(1); \qquad \int_0^1 \rho(r)\,dr=\frac{1}{2} \tag{2.9b}$$

τ_x is the translation (by x) operator, i.e., $\tau_x(\eta)(y)=\eta(x+y)$ for $-x\leqslant y\leqslant L-x$, and $[2rL]$ is the integer part of $2rL$.

Remark. The Burgers equation reads

$$\frac{\partial\rho}{\partial t}+\frac{\partial j}{\partial r}=0$$

with

$$j=-\frac{\partial\rho}{\partial r}+2\theta\rho(1-\rho)$$

By (2.9a), ρ satisfies the stationary Burgers equation with zero current.

We give now a heuristics for the choice of p and for Eq. (2.9). We prove that, in the above limit, $\bar{\mu}_L$ is close to $\nu_{\alpha(L)}$ with $\alpha=\alpha(L)$ chosen as to give total particle density equal to 1/2. That is, α must satisfy the equation

$$\frac{1}{2L}\nu_\alpha\left(\sum_{x=1}^{2L}\eta(x)\right)=\frac{1}{2L}\sum_{x=1}^{2L}\frac{\alpha(p/q)^x}{1+\alpha(p/q)^x}=\frac{1}{2} \tag{2.10}$$

We prove below that (2.10) is equivalent to

$$\nu_\alpha(\eta(1)=1)=\nu_\alpha(\eta(2L)=0) \tag{2.11}$$

which implies

$$\alpha=(p/q)^{-(L+1/2)} \tag{2.12}$$

and

$$\nu_\alpha(\eta([2rL]))=\frac{(p/q)^{[2rL]-(L+1/2)}}{1+(p/q)^{[2rL]-(L+1/2)}} \tag{2.13}$$

For p fixed this will tend to zero if $r<1/2$, and to one if $r>1/2$. In order to obtain a nontrivial limit, one takes ($\theta>0$ and L big enough)

$$p=p(L)=\frac{1}{2}+\frac{\theta}{4L} \tag{2.14}$$

In this case

$$\rho(r)=\lim_{L\to+\infty} \nu_{\alpha(L)}(\eta([2rL]))=\frac{e^{\theta(2r-1)}}{1+e^{\theta(2r-1)}} \tag{2.15}$$

which is the nonconstant solution of Eq. (2.9).

Proof of the Equivalence between (2.10) and (2.11). Notice that, since ν_α is reversible, i.e.,

$$p\nu_\alpha(\eta(x)=1)[1-\nu_\alpha(\eta(x+1)=1)]=q\nu_\alpha(\eta(x+1)=1)[1-\nu_\alpha(\eta(x)=1))]$$

then (2.11) implies that $\nu_\alpha(\eta(x)=1)=\nu_\alpha(\eta(2L-x)=0)$ for all x. Summing up the above equalities, we get (2.10). On the other hand, suppose that (2.11) is false and (say) $\nu_\alpha(\eta(1)=1)>\nu_\alpha(\eta(2L)=0)$; then reversibility implies that $\nu_\alpha(\eta(x)=1)>\nu_\alpha(\eta(2L-x)=0)$ for all x, which contradicts (2.10).

From Theorem 1, and taking account of Eq. (2.7), we obtain the following result about the macroscopic profile of the interface model in equilibrium.

Theorem 2. For $L\geqslant 2$, integer, let $\tilde{\mu}_L$ be the invariant measure for the interface model (ξ_t) on $\mathscr{X}_L$ with $p=p(L)=1/2+\theta/4L$, for some $\theta\in(0,4]$ fixed (cf. Definition 2.1). Then, for any $r\in(0,1)$ and any $\delta>0$

$$\lim_{L\to+\infty} \tilde{\mu}_L\left[\left|\frac{1}{L}\xi^L([2Lr])-R(r)\right|>\delta\right]=0 \tag{2.16}$$

where $R(\cdot)$ is the unique nonconstant solution of

$$R''(r)=\theta\{1-[R'(r)]^2\}, \qquad R(0)=0=R(1) \tag{2.17}$$

3. PROOFS OF THE RESULTS

Theorem 2 follows from Theorem 1 as an easy consequence of relation (2.7). In fact from Eq. (2.8) in Theorem 1 we easily obtain (2.16) with $R(r)=\int_0^r [2\rho(u)-1]\,du$, where $\rho(\cdot)$ is the solution of Eq. (2.9).

Proposition 3.1. Let $\bar{\mu}_L$ and $\rho(\cdot)$ be as in Theorem 1. For $k \geqslant 1$, let $x_i = [r_i 2L] \in \{1,..., 2L\}$, $i = 1,..., k$, where $[\cdot]$ is the integer part. Then

$$\lim_{L \to +\infty} \left| \bar{\mu}_L \left(\prod_{i=1}^{k} \eta(x_i) \right) - \prod_{i=1}^{k} \rho(r_i) \right| = 0 \tag{3.1}$$

Proof. Let us recall that for any $\alpha \in (0, +\infty)$

$$\bar{\mu}_L = \nu_\alpha \left(\cdot \,\middle|\, \sum_{x=1}^{2L} \eta(x) = L \right)$$

where ν_α is the (reversible) product measure given by Eq. (2.5). From the fact of ν_α being product we obtain

$$\bar{\mu}_L \left(\prod_{i=1}^{k} \eta(x_i) \right) = \frac{\nu_\alpha(\sum_{x \neq x_1,...,x_k} \eta(x) = L - k) \prod_{i=1}^{k} \nu_\alpha(\eta(x_i))}{\nu_\alpha(\sum \eta(x) = L)} \tag{3.2}$$

where the sums run over $x \in \{1,..., 2L\}$. Now choosing $\alpha = \alpha(L)$ such that Eq. (2.10) holds, we have, according to Eq. (2.15), that

$$\lim_{L \to +\infty} \nu_\alpha(\eta(x_i)) = \rho(r_i) \tag{3.3}$$

From Eqs. (3.2) and (3.3) the proposition will follow once we prove that

$$\lim_{L \to +\infty} \frac{\nu_\alpha(\sum_{x \neq x_1,...,x_k} \eta(x) = L - k) - \nu_\alpha(\sum_{x=1}^{2L} \eta(x) = L)}{\nu_\alpha(\sum_{x=1}^{2L} \eta(x) = L)} = 0 \tag{3.4}$$

Using Eq. (2.10) and the local central limit theorem (ref. 9, Theorem 5, p. 197), we have

$$\left\{ \sum_{x \neq x_1,...,x_k} \nu_\alpha(\eta(x))[1 - \nu_\alpha(\eta(x))] \right\}^{1/2} \nu_\alpha \left[\sum_{x \neq x_1,...,x_k} \eta(x) = L - k \right]$$
$$= \frac{1}{(2\pi)^{1/2}} \exp \left\{ - \frac{[\sum_{i=1}^{k} \nu_\alpha(\eta(x_i)) - k]^2}{2 \sum_{x \neq x_1,...,x_k} \nu_\alpha(\eta(x))[1 - \nu_\alpha(\eta(x))]} \right\} + O\left(\frac{1}{(2L)^{1/2}} \right) \tag{3.5}$$

and similarly for $k = 0$. Noticing that $(1/L) \sum_{x=1}^{2L} \nu_\alpha(\eta(x))[1 - \nu_\alpha(\eta(x))]$ tends to $\int_0^1 \rho(r)[1 - \rho(r)]\, dr \in (0, +\infty)$, we obtain that there exists a constant $C \in (0, +\infty)$ so that the lhs quotient in the Eq. (3.4) is bounded by $C/\sqrt{L}$. This proves the proposition.

4. CONCLUDING REMARKS

A question arising naturally by Theorem 2 refers to the behavior of the fluctuations of $\xi^L([2rL])/L$ around its deterministic limit. Due to the relation between the interface and the exclusion processes [Eq. (2.7)], this can be obtained from the study of the fluctuation density field of the exclusion process $(\eta^L\cdot)$. That is, one considers the field

$$Y^L(\varphi)=\frac{1}{\sqrt{L}}\sum_{x=1}^{2L}\varphi\left(\frac{x}{2L}\right)[\eta^L(x)-\bar{\mu}_L(\eta(x))] \tag{4.1}$$

where φ is a test function and (η^L) is the (stationary) simple exclusion appearing in Theorem 1. Using techniques very similar to those of refs. 3 and 4, it is possible to prove that the distribution of (Y^L) converges to a Gaussian process. The regular part of the covariance of the limiting process satisfies the linearized Burgers equation

$$\tfrac{1}{2}\partial_r^2 c(r,r')+\tfrac{1}{2}\partial_{r'}^2 c(r,r')+[1-2\rho(r)]c+[1-2\rho(r')]\,c(r,r')$$
$$+\delta(r-r')[-\tfrac{1}{2}\rho'(r)]=0$$

As a consequence, the fluctuations in the interface, $L^{1/2}\{\xi^L([2rL])/L-R(r)\}$, converge in law to a Gaussian limit. We do not work out the details, since the techniques are simpler than the ones used in refs 3 and 4.

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